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Improvement of He's variational iteration method for solving systems of differential equations

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ABSTRACT

In recent years a lot of attention from researchers has been attracted to the various aspects of the well known He's variational iteration method. This method is a very powerful method for solving a large amount of problems. It provides a sequence which converges to the solution of the problem without discretization of the variables. In this work an idea is proposed that accelerates the convergence of the sequences which result from the variational iteration method for solving systems of differential equations. Illustrative examples are presented to show the validity of the new method.

1. Introduction

In this work we consider He's variational iteration method as a well known method for finding both analytical and approximate solutions of systems of differential equations. This technique was developed by the Chinese mathematician He [1]. The variational iteration method is used for solving autonomous ordinary differential systems in [2]. Application of this method to the Helmholtz equation is investigated in [3]. This method is used for solving Burgers' and coupled Burgers' equations in [4]. In [4] the applications of the present method to coupled Schrödinger–KdV equations and shallow water equations are provided. Also the use of this method for solving linear fractional partial differential equations arising from fluid mechanics is presented in [5]. Other recent works in this field are found in [6–8]. The efficiency of this method for solving problems of various types is shown for example in [2–4,9–15]. The variational iteration method is investigated in [28] to solve parabolic integro-differential equations arising in heat conduction in materials with memory. This technique is used in [29] to solve the generalized pantograph equation [30]. The variation iteration procedure is employed in [31] to solve the wave equation subject to an integral conservation condition. This technique is applied in [32] to find the solution of nonlinear mixed Volterra–Fredholm integral equations. This scheme is applied to find the solution of several classes of variational problems [33]. Some recent research works in this field are [16–24].

The organization of the rest of this paper is as follows.

The well known He's variational iteration method is reviewed in Section 2. In Section 3, the use of it for solving systems of differential equations will be discussed and a new method will be proposed for solving these problems. The application of the new idea for solving higher order differential equations will be given in Section 4. To present a clear overview of the method, we select several examples in Section 4. A conclusion is presented in Section 5.

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2. Variational iteration method

In this method, the problems are initially approximated with possible unknowns. Then a corrected functional is constructed using a general Lagrange multiplier, which can be identified optimally via the variational theory [9]. In this method the problem is considered as

$$Ly + Ny = g(x), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. Using the variational iteration method, the following functional is considered:

$$y_{n+1} = y_n + \int_0^x \lambda(Ly_n(s) + N\tilde{y}_n(s) - g(s))\mathrm{d}s, \tag{2.2}$$

where λ is a Lagrange multiplier, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$ [25–27]. Taking the variation from both sides of the correct functional with respect to y_n and imposing $\delta y_{n+1} = 0$, the stationary conditions are obtained. Using the stationary conditions the optimal value of λ is identified.

Since this method avoids the discretization [34] of the problem, it is possible to find a closed form solution without any round-off error. The use of symbolic computation is necessary for finding the iterations.

3. The system of differential equations

In the case of the m equations, we rewrite equations in the form

$$L_i(y_i) + N_i(y_1, \ldots, y_m) = g_i(x), \quad i = 1, \ldots, m, \tag{3.1}$$

where L_i is linear with respect to y_i, and N_i is the nonlinear part of the ith equation. In this case the functionals are obtained as

$$y_{i(n+1)} = y_{in} + \int_0^x \lambda_i(L_i(y_{in}(s)) + N(\tilde{y}_{1n}(s), \ldots \tilde{y}_{mn}(s)) - g(s))\mathrm{d}s, \tag{3.2}$$

and the optimal values of λ_i, $i = 1, \ldots, m$, are obtained by taking the variation from both sides of the functionals and finding stationary conditions using

$$\delta y_{i(n+1)} = 0, \quad i = 1 \ldots m.$$

Our main idea in this paper is the use of the following system of sequences instead of the system which results from the variational iteration method:

$$y_{i(n+1)} = y_{in} + \int_0^x \lambda_i(L_i(y_{in}(s)) + N(y_{1(n+1)}(s), \ldots, y_{(i-1)(n+1)}, y_{i(n)}, \ldots, y_{mn}(s)) - g(s))\mathrm{d}s, \tag{3.3}$$

for $i = 2, \ldots, m$. In fact the updated values $y_{1(n+1)}, y_{2(n+1)}, \ldots, y_{(i-1)(n+1)}$ are used for finding $y_{i(n+1)}$. This technique accelerates the convergence of the system of sequences. Therefore, using just a few terms of the sequences, an accurate solution can be obtained for a larger domain of the problem. It is especially useful when computing more terms of the sequences is difficult or impossible. The effect of this correction is clear in $y_{m(n+1)}$ because the updated values are used to compute it.

4. Higher order differential equations

As an application of the new method we emphasize the solution of higher order differential equations. Consider the following differential equations of order n:

$$y^{(n)} + f(y, y', \ldots, y^{n-1}) = g(x), \tag{4.1}$$

with appropriate initial conditions. It is clear that finding the Lagrange multiplier is not easy for this equation especially when n is large. So it is better to transform it to a first-order system of differential equations as follows:

$$y_1 = y', \tag{4.2}$$

$$y_2 = y_1', \tag{4.3}$$

$$\vdots$$

$$y_n' + f(y, y_1, \ldots, y_{(n-1)}) = g(x). \tag{4.4}$$

The above system can be solved using the He's variational method. Also it is possible to use the new method on this system of equations. An important point for solving this system of equations using the new method is the arrangement of the

Table 1
The results from the variational iteration method (VIM) and the corrected variational iteration method (CVIM) in Example 1.

t	$\lvert x(t) - x_{10}(t)\rvert$	$\lvert x(t) - \bar{x}_{10}(t)\rvert$	$\lvert y(t) - y_{10}(t)\rvert$	$\lvert y(t) - \bar{y}_{10}(t)\rvert$
0.1	0.3189×10^{-11}	0.2987×10^{-14}	0.4253×10^{-11}	0.1173×10^{-14}
0.2	0.6625×10^{-8}	0.1072×10^{-10}	0.8834×10^{-8}	0.4741×10^{-11}
0.3	0.5814×10^{-6}	0.1520×10^{-8}	0.7752×10^{-6}	0.7390×10^{-9}
0.4	0.1396×10^{-4}	0.5618×10^{-7}	0.1862×10^{-4}	0.2956×10^{-7}
0.5	0.1650×10^{-3}	0.9827×10^{-6}	0.2200×10^{-3}	0.5543×10^{-6}
0.6	0.1244×10^{-2}	0.1064×10^{-4}	0.1659×10^{-2}	0.6385×10^{-5}
0.7	0.6888×10^{-2}	0.8255×10^{-4}	0.9185×10^{-2}	0.5232×10^{-4}
0.8	0.3039×10^{-1}	0.4996×10^{-3}	0.4052×10^{-1}	0.3330×10^{-3}
0.9	0.1127	0.2498×10^{-2}	0.1503	0.1744×10^{-2}

resulted sequences. As we know, there are $n!$ arrangements for n sequences. Here a question arises: What is the most suitable arrangement of sequences? The answer to the question is

$$y_{n(k+1)} = y_{nk} - \int_0^\tau (y'_{nk} + f(y_k, y_{1k}, \ldots, y_{(n-1)k}) - g(\tau))\mathrm{d}\tau, \tag{4.5}$$

$$\vdots$$

$$y_{1(k+1)} = y_{1k} - \int_0^\tau (y'_{1(k)} - y_{2(k+1)})\mathrm{d}\tau, \tag{4.6}$$

$$y_{k+1} = y_k - \int_0^\tau (y'_{k+1} - y_{1(k+1)k})\mathrm{d}\tau. \tag{4.7}$$

The reason for making this choice is to use more updated values for finding y_{k+1}. In fact in this arrangement y_{k+1} and its derivatives are found using the newest possible values.

5. Illustrative examples

5.1. Example 1

As an elementary example we consider the following system of two differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 3y - 3\exp(2t), \qquad x(0) = 1, \tag{5.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x + 2y - 4\exp(t), \qquad y(0) = 1. \tag{5.2}$$

The exact solution of this problem is $x(t) = \exp(t)$ and $y(t) = \exp(2t)$. Using the variational iteration method we find the following sequences:

$$x_{n+1} = x_n - \int_0^t (x'_n - x_n - 3y_n + 3\exp(2\tau))\mathrm{d}\tau, \qquad x_0 = 1, \tag{5.3}$$

$$y_{n+1} = y_n - \int_0^t (y'_n - 4x_n - 2y_n - 4\exp(\tau))\mathrm{d}\tau, \qquad y_0 = 1. \tag{5.4}$$

The use of the modified method results in the sequences

$$\bar{x}_{n+1} = \bar{x}_n - \int_0^t (\bar{x}'_n - \bar{x}_n - 3\bar{y}_n + 3\exp(2\tau))\mathrm{d}\tau, \qquad \bar{x}_0 = 1, \tag{5.5}$$

$$\bar{y}_{n+1} = \bar{y}_n - \int_0^t (\bar{y}'_n - 4\bar{x}_{n+1} - 2\bar{y}_n - 4\exp(\tau))\mathrm{d}\tau, \qquad \bar{y}_0 = 1. \tag{5.6}$$

In Table 1, the results from VIM and CVIM are shown. It is clear from this table that the use of the new method provides a more accurate solution in a larger domain. Also in Fig. 1, the error functions $x(t) - x_{15}(t)$, $x(t) - \bar{x}_{15}(t)$, $y(t) - y_{15}(t)$ and $y(t) - \bar{y}_{15}(t)$ are plotted. The results show the improvement of the method.

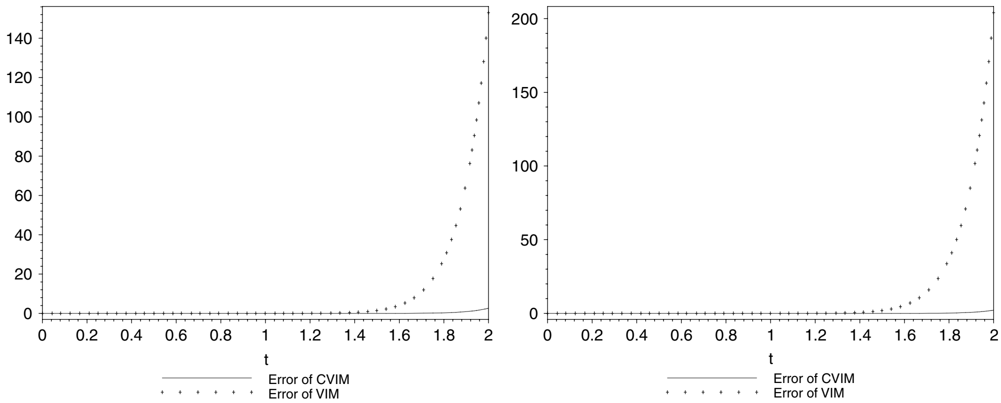

Fig. 1. Plot of the error function for $x_{15}(t)$, $\bar{x}_{15}(t)$ (left) and the error function for $y_{15}(t)$, $\bar{y}_{15}(t)$ (right).

Notice that the system of ordinary differential equations (5.1)–(5.2) can be written as the following system of integral equations:

$$x(t) = \int_0^t x(\tau)\mathrm{d}\tau + 3\int_0^t y(\tau)\mathrm{d}\tau - \frac{3}{2}\exp(2t) + \frac{5}{2}, \tag{5.7}$$

$$y(t) = 4\int_0^t x(\tau)\mathrm{d}\tau + 2\int_0^t y(\tau)\mathrm{d}\tau - 4\exp(t) + 5. \tag{5.8}$$

Therefore, according to the results obtained, we can conclude that the new method is also efficient for solving systems of integral equations.

5.2. Example 2

In this example, we consider the following third-order differential equation:

$$y'''(t) - 2y''(t) + 2y'(t) = 0, \qquad y(0) = -2, \qquad y'(0) = 0, \qquad y''(0) = -4. \tag{5.9}$$

The exact solution of this problem is $-2 - 2\exp(t)\sin(t) + 2\exp(t)\cos(t)$. Generally finding the Lagrange multiplier is not easy for higher order differential equations. These equations should be changed to a system of differential equations. In this case we have

$$y' = x, \qquad y(0) = -2, \tag{5.10}$$

$$x' = z, \qquad x(0) = 0, \tag{5.11}$$

$$z' - 2z + 2x = 0, \qquad z(0) = -4. \tag{5.12}$$

Therefore the new method is useful for solving the resulting system. In this example the use of the variational iteration method leads to

$$y_{n+1} = y_n - \int_0^t (y_n' - x_n)\mathrm{d}\tau, \qquad y_0 = -2, \tag{5.13}$$

$$x_{n+1} = x_n - \int_0^t (x_n' - z_n)\mathrm{d}\tau, \qquad x_0 = 0, \tag{5.14}$$

$$z_{n+1} = z_n - \int_0^t (z_n' - 2z_n + 2x_n)\mathrm{d}\tau, \qquad z_0 = -4. \tag{5.15}$$

The three sequences can be arranged in 3! ways. As we said in Section 4, the best one for implementation of the new method is

$$\bar{z}_{n+1} = \bar{z}_n - \int_0^t (\bar{z}_n' - 2\bar{z}_n + 2\bar{x}_n)\mathrm{d}\tau, \qquad z_0 = -4, \tag{5.16}$$

$$\bar{x}_{n+1} = \bar{x}_n - \int_0^t (\bar{x}_n' - \bar{z}_{n+1})\mathrm{d}\tau, \qquad x_0 = 0, \tag{5.17}$$

$$\bar{y}_{n+1} = \bar{y}_n - \int_0^t (\bar{y}_n' - \bar{x}_{n+1})\mathrm{d}\tau, \qquad y_0 = -2. \tag{5.18}$$

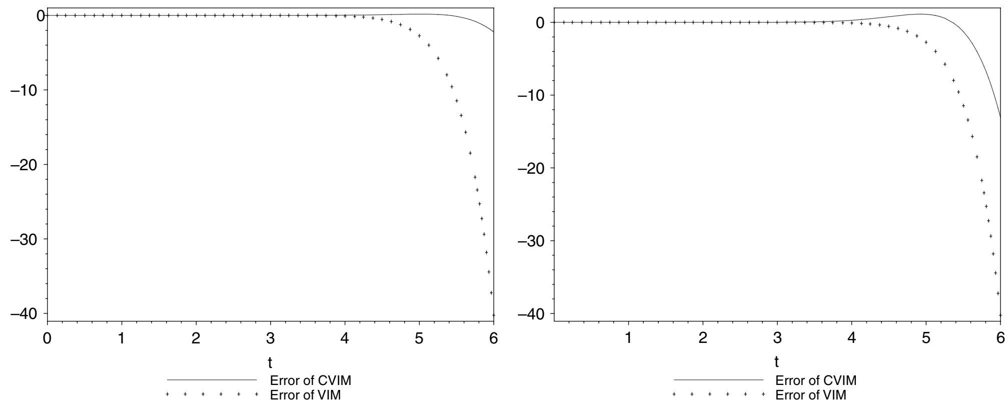

Fig. 2. Plot of the error functions $y(t) - y_{15}(t), y(t) - \bar{y}_{15}(t)$ (left) and $y(t) - y_{15}(t), y(t) - \tilde{y}_{15}(t)$ (right).

As another arrangement of sequences for use of the new method consider

$$\tilde{y}_{n+1} = \tilde{y}_n - \int_0^t (\tilde{y}_n' - \tilde{x}_n)\mathrm{d}\tau, \qquad y_0 = -2, \tag{5.19}$$

$$\tilde{x}_{n+1} = \tilde{x}_n - \int_0^t (\tilde{x}_n' - \tilde{z}_n)\mathrm{d}\tau, \qquad x_0 = 0, \tag{5.20}$$

$$\tilde{z}_{n+1} = \tilde{z}_n - \int_0^t (\tilde{z}_n' - 2\tilde{z}_n + 2\tilde{x}_{n+1})\mathrm{d}\tau, \qquad z_0 = -4. \tag{5.21}$$

In Fig. 2, the error functions $y(t) - y_{15}(t), y(t) - \bar{y}_{15}(t)$ and $y(t) - y_{15}(t), y(t) - \tilde{y}_{15}(t)$ are plotted.

5.3. Example 3

In this example consider the following system of partial differential equations:

$$z_t = u_{xx}, \tag{5.22}$$

$$u_t = z, \tag{5.23}$$

for $x \in \mathbb{R}$. The corresponding initial conditions are given as follows:

$$z(x, 0) = 0, \tag{5.24}$$

$$u(x, 0) = \frac{1}{8}\sin(\pi x). \tag{5.25}$$

The exact solution of this problem is

$$z(x, t) = -\frac{\pi}{8}\sin(\pi x)\sin(\pi t), \tag{5.26}$$

$$u(x, t) = \frac{1}{8}\sin(\pi x)\cos(\pi t). \tag{5.27}$$

In fact this system presents the wave equation

$$u_{tt} = u_{xx}, \tag{5.28}$$

for $x \in \mathbb{R}$ with initial conditions

$$u(x, 0) = \frac{1}{8}\sin(\pi x), \tag{5.29}$$

$$u_t(x, 0) = 0. \tag{5.30}$$

The well known He's variational iteration method provides the following sequences:

$$z_{n+1} = z_n - \int_0^t ((z_t)_n - (u_{xx})_n)\mathrm{d}\tau, \qquad z_0 = 0, \tag{5.31}$$

$$u_{n+1} = u_n - \int_0^t ((u_t)_n - z_n)\mathrm{d}\tau, \qquad u_0 = \frac{1}{8}\sin(\pi x). \tag{5.32}$$

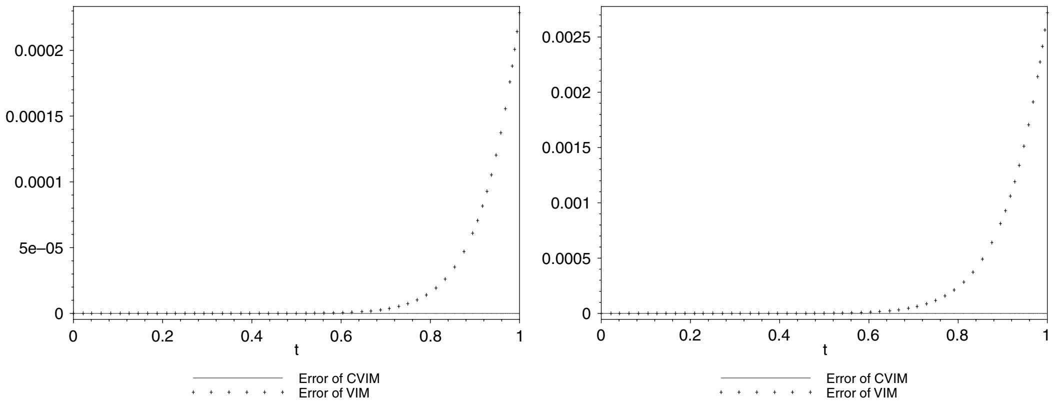

Fig. 3. Plot of the error functions $u(0.5,t)-u_{10}(0.5,t)$ and $u(0.5,t)-\bar{u}_{10}(0.5,t)$ (left) and $z(0.5,t)-z_{10}(0.5,t)$ and $z(0.5,t)-\bar{z}_{10}(0.5,t)$ (right).

Also the new method suggests the system of sequences

$$\bar{z}_{n+1}=\bar{z}_n-\int_0^t((\bar{z}_t)_n-(\bar{u}_{xx})_n)\mathrm{d}\tau,\qquad \bar{z}_0=0,\tag{5.33}$$

$$\bar{u}_{n+1}=\bar{u}_n-\int_0^t((\bar{u}_t)_n-\bar{z}_{n+1})\mathrm{d}\tau,\qquad \bar{u}_0=\frac{1}{8}\sin(\pi x).\tag{5.34}$$

In Fig. 3, the error functions $u(0.5,t)-u_{10}(0.5,t)$, $u(0.5,t)-\bar{u}_{10}(0.5,t)$, $z(0.5,t)-z_{10}(0.5,t)$ and $z(0.5,t)-\bar{z}_{10}(0.5,t)$ are plotted.

The technique developed in this paper can be employed to determine the solution of the second Painleve equation investigated in [35]. As another research proposal we can mention the use of new method to solve the nonlinear systems of Volterra integro-differential equations with delay arguments studied in [36].

6. Conclusion

The variational iteration method is an efficient method for solving various kinds of problems. In this paper He's variational iteration method is modified for solving differential systems of equations. In fact the main idea is the use of the new values of the computed variables in a special stage for finding the new value of the corresponding variable in this stage. Acceleration of the convergence of the system of sequences to the exact solution of the problem and extension of the domain on which the approximate solution is acceptable are the advantages of the new method. Also this method is implemented for solving higher order ordinary differential equations by transforming the problem to a system of first-order differential equations. It is mentioned that the arrangements of the equations are important in the use of this method. The new method can be used for solving systems of partial differential equations, integral equations, integro-differential equations etc.

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